Yang-Mills Theory for Noncommutative Flows
Addendum

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Abstract

This supplementary manuscript is to describe an important nontrivial example, which appears in the matrix model of type IIB in the super string theory in order to apply a new duality for the moduli spaces of Yang-Mills connections on noncommutative vector bundles. Actually, the moduli space of the instanton bundle over noncommutative Euclidean 4-spaces with respect to the canonical action of space translations is computed precisely without using the ADHM- construction.

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§1. Introduction In the manuscript [6], we have found a new duality for the moduli spaces of Yang-Mills connections on noncommutative vector bundles with respect to noncommutative flows. As we have also announced in [6] with a short proof that such a duality was also affirmatively shown for noncommutative multiflows. In this addendum, we apply it to compute the moduli space in the case of the instanton bundles on the noncommutative Euclidean 4-space with respect to the canonical space translations without using the ADHM construction (cf:[1],[2]).

§2. Preliminaries Let $(A, \mathbb{R}^n, \alpha)$, $(n \geq 1)$ be a F^*-system, and Ξ a finitely generated projective α-invariant right A-module. As $\Xi = P(A^m)$ for a projection $P \in M_m(A)$ over A $(m \geq 1)$, and let $\widehat{\Xi} = \widehat{P}(\widehat{A}^m)$ where $\widehat{A} = A \rtimes_\alpha \mathbb{R}^n$ and $\widehat{P} = P \times I \in M_m(\mathcal{M}(\widehat{A}))$ where $\mathcal{M}(\widehat{A})$ is the F^*-algebra consisting of all A-valued bounded C^∞-functions on $\mathbb{R}^n$ with α-twisted *-convolution. Then $\widehat{\Xi}$ is a finitely generated projective right $\widehat{A}$-module. In [6], we have shown the following theorem:

Theorem 1 ([6]). Let $(A, \mathbb{R}^n, \alpha)$ be a F^*-dynamical system with a faithful α-invariant continuous trace τ, and Ξ a finitely generated projective right A-module. Then there exist a dual F^*-dynamical system $(\widehat{A}, \mathbb{R}^n, \overline{\alpha})$ with a faithful $\overline{\alpha}$-invariant trace $\widehat{\tau}$ and a finitely generated projective right $\widehat{A}$-module $\widehat{\Xi}$ with the property that the moduli space $\mathcal{M}^{(A,\mathbb{R}^n,\alpha,\tau)}(\Xi)$ of the Yang-Mills connections of Ξ for $(A, \mathbb{R}^n, \alpha, \tau)$ is homeomorphic to the dual moduli space $\mathcal{M}^{(\widehat{A},\mathbb{R}^n,\overline{\alpha},\widehat{\tau})}(\widehat{\Xi})$ of the Yang-Mills connections of $\widehat{\Xi}$ for $(\widehat{A}, \mathbb{R}^n, \overline{\alpha}, \widehat{\tau})$.

Theorem 2 ([6]). Let $(\widehat{A}, \mathbb{R}^n, \beta)$ be a F^*-dynamical system with a faithful β-invariant continuous trace τ,and Ξ a finitely generated projective right $\widehat{A}$-module. If β commutes with $\overline{\alpha}$, then there exist a F^*-dynamical system $(A, \mathbb{R}^n, \beta_A)$ with a faithful β_A-invariant continuous trace τ_A, and a finitely generated projective right A-module Ξ_A such that

$$\mathcal{M}^{(\widehat{A}, \mathbb{R}^n, \beta, \tau)}(\Xi) \approx \mathcal{M}^{(A, \mathbb{R}^n, \beta_A, \tau_A)}(\Xi_A) \ .$$

We want to apply the above theorem to the following important example which appears as a Higgs branch of the theory of D0-branes bound to D4-branes by the expectation value of the B-field as well as a regularized version of the target space of supersymmetric quantum mechanics arising in the light cone description of (2,0) superconformal theories in six dimensions, although its algebraic structure has already been established in the example 10.1 of [5](cf.[3],[4]):

Let $\mathbb{R}^4_\theta$ be the noncommutative $\mathbb{R}^4$ for an antisymmetric 4x4 matrix $\theta = (\theta_{i,j})$, in other words, $\mathbb{R}^4_\theta$ is the F^*-algebra generated by 4-self adojoint elements $\{x_i\}_{i=1}^4$ with the property that

$$[x_i, x_j] = \theta_{i,j} \tag{1}$$

$(i, j = 1, \cdots, 4)$. In other words,

$$\mathbb{R}^4_\theta \ = \ \{ \sum_{i_1,i_2,i_3,i_4 \in \mathbb{N}} c_{i_1,i_2,i_3,i_4} \ x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4} \mid c \in S(\mathbb{N}) \ \}$$

where $S(\mathbb{N})$ is the set of all rapidly decreasing complex valued functions on $\mathbb{N}$. Let $x^i = \theta^{i,j} x_j$ $(i, j = 1, \cdots, 4)$ where $(\theta^{i,j})$ is the inverse matrix of $(\theta_{i,j})$. Then the F^*-

algebra $\mathbb{R}^4_\theta$ depends essentially on one positive real number denoted by the same symbol θ, which satisfy the following relation:

$$(2)\quad [z_i^*, z_i] = \theta\ ,\ [z_i, z_j] = [z_i^*, z_j] = 0\quad (i, j = 0, 1, i \neq j)$$

where $z_0 = x^1 + \sqrt{-1}x^2, z_1 = x^3 + \sqrt{-1}x^4$ and z_i^* are the conjugate operators of z_i. Let us consider the canonical action α of $\mathbb{R}^4$ on $\mathbb{R}^4_\theta$ defined by

$$(3)\qquad \alpha_{t_i}(x_i) = x_i + t_i$$

($t_i \in \mathbb{R}, i = 1, \cdots, 4$). Then it is easily seen that the triplet $(\mathbb{R}^4_\theta, \mathbb{R}^4, \alpha)$ is a F^*-dynamical system, and we easily see that

$$(4)\qquad \alpha_{w_i}(z_i) = z_i + w_i$$

($w_i \in \mathbb{C}, i = 0, 1$). By (2), $\mathbb{R}^4_\theta$ is nothing but the F^*-tensor product $A_0 \otimes A_1$ where A_i are the F^*-algebras generated by z_i ($i = 0, 1$). We now check the algebraic structure of A_i. By (2), it follows from [3](cf.[4]) that there exist two Fock spaces H_i such that

$$z_i(\xi^i_n) = \sqrt{(n+1)\theta}\ \xi^0_{n+1}\ ,\ z_i^*(\xi^i_n) = \sqrt{n\theta}\ \xi^i_{n-1},$$

where $\{\xi^i_n\}$ are complete orthonormal systems of H_i with respect to the following inner product:

$$<\ f \mid g\ >= \sum (n+1)\theta\ f(n)\overline{g(n)}$$

for two $\mathbb{C}$-valued functions f, g on $\mathbb{N}$ such that

$$\sum (n+1)\theta\ |f(n)|^2 < \infty\ ,\ \sum (n+1)\theta\ |g(n)|^2 < \infty\ .$$

for $i = 0, 1$. We may assume that the A_i act on H_i irreducibly. Then it also follows from [4] that the F^*-algebras A_i are isomorphic to the F^*-algebras $\mathcal{K}^\infty(H_i)$ defined by

$$\mathcal{K}^\infty(H_i) = \{T \in \mathcal{K}(H_i) \mid \{\lambda_k\} \in S(\mathbb{N})\}$$

where $\{\lambda_k\}$ are all eigen values of T and $S(\mathbb{N})$ are the set of all sequences $\{c_n\}$ of $\mathbb{C}$ with $sup_{n\geq 1}\,(1+|n|)^k|c_n| < \infty$ for all $k \geq 0$. Therefore, the F^*-algebra $\mathbb{R}^4_\theta$ is isomorphic to $\mathcal{K}^\infty(H_0 \otimes H_1)$. We then have the following proposition:

Proposition 3 (cf:[5]). If $\theta \neq 0$, then $\mathbb{R}^4_\theta$ is isomorphic to $\mathcal{K}^\infty(L^2(\mathbb{C}^2))$ as a F^*-algebra.

By the above Proposition, $\mathcal{K}^\infty(L^2(\mathbb{C}^2))$ is the F^*-crossed product $S(\mathbb{C}^2) \rtimes_\tau \mathbb{C}^2$ of $S(\mathbb{C}^2)$ by the shift action τ of $\mathbb{C}^2$. We then consider the action α defined before. By (4), it follows from [R] that α plays a role of the dual action of τ. Then the F^*-crossed product $\widehat{\mathbb{R}^4_\theta}$ of $\mathbb{R}^4_\theta$ by the action α of $\mathbb{R}^4$ is isomorphic to the F^*-crossed product $\mathcal{K}^\infty(L^2(\mathbb{C}^2)) \rtimes_{\widehat{\tau}} \mathbb{C}^2$, where $\widehat{\tau}$ is the dual action of τ. Then it is isomorphic to $S(\mathbb{C}^2) \otimes \mathcal{K}^\infty(L^2(\mathbb{C}^2))$ as a F^*-algebra. We now consider a finitely generated projective right $\mathbb{R}^4_\theta$-module Ξ. Then there exist an integer $n \geq 1$ and a projection $P \in M_n(\mathcal{M}(\mathbb{R}^4_\theta))$ such that $\Xi = P((\mathbb{R}^4_\theta)^n)$. where $\mathcal{M}(\mathbb{R}^4_\theta)$ is the F^*-algebra consisting of all bounded linear operators T on $L^2(\mathbb{C}^2)$ whose kernel functions $T(\cdot\,,\ \cdot)$ are $\mathbb{C}$-valued bounded C^∞-functions of $\mathbb{C}^2 \times \mathbb{C}^2$. Let us take the canonical faithful trace Tr on $\mathbb{R}^4_\theta$ because of Proposition 1. Then we consider the moduli space:

$\mathcal{M}^{(\mathcal{K}^\infty(L^2(\mathbb{C}^2)),\mathbb{C}^2,\alpha,Tr)}(\Xi)$ of Ξ for $(\mathcal{K}^\infty(L^2(\mathbb{C}^2)), \mathbb{C}^2, \alpha, Tr)$.

We want to describe P cited above as a precise fashion. Actually, we know that

$$\mathcal{K}^\infty(L^2(\mathbb{C}^2)) \cong S(\mathbb{C}^2) \rtimes_\lambda \mathbb{C}^2$$

where $\cong$ means isomorphism as a F^*-algebra. λ is the shift action of $\mathbb{C}^2$ on $S(\mathbb{C}^2)$. Then it follows that

$$M_n(\mathcal{K}^\infty(L^2(\mathbb{C}^2))) \cong M_n(S(\mathbb{C}^2)) \rtimes_{\lambda^n} \mathbb{C}^2$$

where

$$\lambda_w^n(f)(w') = f(w' - w)$$

for $f \in M_n(S(\mathbb{C}^2)), w, w' \in \mathbb{C}^2$. Let $\overline{\lambda^n}$ be the action of $\mathbb{C}^2$ on $M_n(\mathcal{K}^\infty(L^2(\mathbb{C}^2)))$ associated with λ^n satisfying Theorem 2. It follows from Proposition 3 that

$$\mathcal{M}^{(\mathbb{R}^4_\theta, \mathbb{R}^4, \alpha, Tr)}(\Xi) \approx \mathcal{M}^{(\mathcal{K}^\infty(L^2(\mathbb{C}^2), \mathbb{C}^2, \alpha, Tr)}(\Xi_1)$$

$$\approx \mathcal{M}^{(S(\mathbb{C}^2)\rtimes_\lambda \mathbb{C}^2, \mathbb{C}^2, \widehat{\lambda}, \widehat{\int_{\mathbb{C}^2} dz})}(\Xi_2) \ ,$$

where

$$\Xi_1 = P_1(\mathcal{K}^\infty(L^2(\mathbb{C}^2)^n) \ , \ \Xi_2 = P_2((S(\mathbb{C}^2) \rtimes_\lambda \mathbb{C}^2)^n)$$

for the two projections P_j $(j = 1, 2)$ with the property that

$$P_1 \in M_n(\mathcal{M}(\mathcal{K}^\infty(L^2(\mathbb{C}^2))) \ , \ P_2 \in M_n(\mathcal{M}(S(\mathbb{C}^2)) \rtimes_\lambda \mathbb{C}^2)$$

corresponding to Ξ, where $\mathcal{M}(S(\mathbb{C}^2)$ is the F^*-algebra consisting of all $\mathbb{C}$-valued bouded C^∞-functions on $\mathbb{C}^2$ and λ is the shift action of $\mathbb{C}^2$ on $\mathcal{M}(S(\mathbb{C}^2))$. By its definition, we know that

$$\overline{\lambda}_w = \widehat{\lambda}_w \circ \widetilde{\lambda}_w \ , \ (w \in \mathbb{C}^2)$$

where $\widehat{\lambda}$ is the dual action of λ and

$$\widetilde{\lambda}_w(x)(w') = \lambda_w x(w')$$

for all $x \in S(\mathbb{C}^2) \rtimes_\lambda \mathbb{C}^2$ and $w, w' \in \mathbb{C}^2$. Hence $\widehat{\lambda}$ commutes with $\overline{\lambda}$, which implies by Theorem 2 that there exist a F^*-dynamical system $(S(\mathbb{C}^2), \mathbb{C}^2, \widehat{\lambda}_{S(\mathbb{C}^2)}, \int_{\mathbb{C}^2} dz)$ and a finitely generated projective right A-module $(\Xi_2)_{S(\mathbb{C}^2)}$ such that

$$\mathcal{M}^{(S(\mathbb{C}^2)\rtimes_\lambda \mathbb{C}^2, \mathbb{C}^2, \widehat{\lambda}, \widehat{\int_{\mathbb{C}^2} dz})}(\Xi_2)$$

$$\approx \quad \mathcal{M}^{(S(\mathbb{C}^2), \mathbb{C}^2, \widehat{\lambda}_{S(\mathbb{C}^2)}, \int_{\mathbb{C}^2} dz)}((\Xi_2)_{S(\mathbb{C}^2)}) \ .$$

We know that there exist an integer $m \geq 1$ and a projection $Q \in M_m(\mathcal{M}(S(\mathbb{C}^2)))$ such that

$$(\Xi_2)_{S(\mathbb{C}^2)} = Q(S(\mathbb{C}^2)^m) \ .$$

Moreover, it follows from the definition that the action $\widehat{\lambda}_{S(\mathbb{C}^2)}$ is nothing but λ. We now determine the moduli space $\mathcal{M}^{(S(\mathbb{C}^2), \mathbb{C}^2, \lambda, \int_{\mathbb{C}^2} dz)}(Q(S(\mathbb{C}^2)^m)$ in what follows: Since $Q \in M_m(\mathcal{M}(S(\mathbb{C}^2)))$ and

$$M_m(S(\mathbb{C}^2)) \cong S(\mathbb{C}^2, M_m(\mathbb{C})) \ ,$$

then it also follows from Theorem 2 that there exist a finitely generated projective right $\mathbb{C}$-module $Q(S(\mathbb{C}^2)^m)_{\mathbb{C}}$ such that

$$\mathcal{M}^{(S(\mathbb{C}^2), \mathbb{C}^2, \widehat{\lambda}_{S(\mathbb{C}^2)}, \int_{\mathbb{C}^2} dz)}(Q(S(\mathbb{C}^2)^m))$$

$$\approx \quad \mathcal{M}^{(\mathbb{C}, \mathbb{C}^2, \lambda_{\mathbb{C}}, 1)}(Q(S(\mathbb{C}^2)^m)_{\mathbb{C}}) \ .$$

Since $Q(S(\mathbb{C}^2)^m)_{\mathbb{C}}$ is a finitely generated projective right $\mathbb{C}$-module, then its construction tells us that there exists a projection $R \in M_m(\mathbb{C})$ such that

$$Q(S(\mathbb{C}^2)^m)_{\mathbb{C}}) \ = \ R(\mathbb{C}^m) \ .$$

Summing up the argument discussed above, we deduce that

$$\mathcal{M}^{(\mathbb{R}^4_\theta,\mathbb{R}^4,\alpha,Tr)}(\Xi) \approx \mathcal{M}^{(\mathbb{C},\mathbb{C}^2,\iota,1)}(R(\mathbb{C}^m)) \ .$$

By the definition of the moduli space, we deduce that

$$\mathcal{M}^{(\mathbb{C},\mathbb{C}^2,\iota,1)}(R(\mathbb{C}^m))$$

$$\approx \mathrm{End}_{\mathbb{C}}(R(\mathbb{C}^m))_{sk}/U(\mathrm{End}_{\mathbb{C}}(R(\mathbb{C}^m)) \ ,$$

where

$$\mathrm{End}_{\mathbb{C}}(R(\mathbb{C}^m))_{sk} \quad (\text{resp. } U(\mathrm{End}_{\mathbb{C}}(R(\mathbb{C}^m))))$$

is the set of all skew adjoint (resp. unitary) elements in $\mathrm{End}_{\mathbb{C}}(R(\mathbb{C}^m))$. Since $\mathrm{End}_{\mathbb{C}}(R(\mathbb{C}^m)) = M_k(\mathbb{C})$ for some natural number k $(m \geq k)$, it follows by using diagonalization that

$$\mathrm{End}_{\mathbb{C}}(R(\mathbb{C}^m))_{sk}/U(\mathrm{End}_{\mathbb{C}}(R(\mathbb{C}^m)) \approx \mathbb{R}^k \ ,$$

which implies the following theorem:

Theorem 4. Let $\mathbb{R}^4_\theta$ be the deformation quantization of $\mathbb{R}^4$ with respect to a skew symmetric matrix θ and take the F^*-dynamical system $(\mathbb{R}^4_\theta, \mathbb{R}^4, \alpha)$ with a canonical faithful α-invariant trace Tr of $\mathbb{R}^4_\theta$, where α is the translation action of $\mathbb{R}^4$ on $\mathbb{R}^4_\theta$. Suppose Ξ is a finitely generated projective right $\mathbb{R}^4_\theta$-module, then there exists a natural number k such that

$$\mathcal{M}^{(\mathbb{R}^4_\theta,\mathbb{R}^4,\alpha,Tr)}(\Xi) \approx \mathbb{R}^k \ .$$

Remark. The above theorem only states the topological data of the moduli spaces of Yang-Mills connections. We would study their both differential and holomorphic structures in a forthcoming paper (cf:[2]).

References

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